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On the Stability of the $O(N)$-Invariant and the Cubic-Invariant Three-Dimensional N-Component Renormalization-Group Fixed Points in the Hierarchical Approximation

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We compute renormalization-group fixed points and their spectrum in an ultralocal approximation. We study a case of two competing nontrivial fixed points for a three-dimensional real N-component field: the $O(N)$-invariant fixed point vs. the cubic-invariant fixed point. We compute the critical value N_c of the cubic ϕ^4-perturbation at the $O(N)$-fixed point. The $O(N)$-fixed point is stable under a cubic ϕ^4-perturbation below N_c; above N_c it is unstable. The Critical value comes out as $2.219435 < N_c < 2.219436$ in the ultralocal approximation. We also compute the critical value of N at the cubic invariant fixed point. Within the accuracy of our computations, the two values coincide.

KEY WORDS: Renormalization group; fixed points; cubic invariance.

1. INTRODUCTION

Spin systems with an N-component real field variable, governed by a Landau–Ginzburg type Hamiltonian, are of central importance both in Euclidean quantum field theory and classical statistical mechanics. The basic model is one with a global $O(N)$ symmetry, the invariance under a simultaneous rotation of all the spins. A prototypical Hamiltonial for such a model, say on a lattice $\Lambda \subset \mathbb{Z}^D$, is

$$H = \sum_{x \in \Lambda} \left\{ \frac{1}{2} \sum_{a=1}^{N} \left(\sum_{\mu=1}^{D} \partial_\mu \phi_a(x)^2 + \phi_a(x)^2 \right) + \lambda \left(\sum_{a=1}^{N} \phi_a(x)^2 \right)^2 \right\} \tag{1}$$

It is important to study the influence of perturbations which explicitely break this symmetry to a smaller subgroup. For instance, in cubical crystals,

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one expects the spin interaction to react to the lattice structure. This suggests additional terms in the Hamiltonian which are not rotation symmetric, but invariant under the cubic group. The cubic group is composed of the permutations and reflections of the N components of the field. Such a cubic invariant term is

$$\mu \sum_{x \in \Lambda} \sum_{a=1}^{N} \phi_a(x)^4 \tag{2}$$

When (1) is augmented by (2), a competition of renormalization group fixed points sets in, about which of them determines the long distance behaviour of the model.

According to folklore, the $O(N)$ fixed point becomes unstable above a certain threshold value N_c. When this happens, the cubical fixed point dominates the long distance behaviour. The value of N_c is still being debated. Early work on the ε-expansion suggests that $N_c > 3$. This has been challenged by studies of Maier, Sokolov, and Shalaev[13, 14] and by Shpot,[17] which suggest that $N_c < 3$. Recent Monte Carlo work[3] by Caselle and Hasenbusch again indicates that N_c should be very close to 3. Within their precision, this is compatible with the picture from recent high order perturbation expansions by Kleinert *et al.*[7–9] and by Shalaev, Antonenko, and Sokolov.[12]

To our knowledge, the problem has not yet been looked at in the framework of the block spin renormalization group.[18] To study the domains of attractions in a model with several competing fixed points is a fundamental and challenging problem as it underlies Wilson's explanation of universality. A notorious trouble in this business is that one has to consider general effective Hamiltonians which depend on an infinite number of interaction terms. To control flows with a large number of couplings is a very difficult task. Furthermore, when symmetries are reduced, the number of couplings tends to proliferate dramatically.

This paper contains a study of both the $O(N)$-invariant fixed point and the cubic fixed point in the framework of the hierarchical or ultralocal approximation to Wilson's renormalization group. Here we restrict our attention to the case of $D = 3$ dimensions. In the hierarchical approximation, the effective Hamiltonians are restricted to a (non-standard) kinetic term plus local interactions. The local interactions are given by a potential, which is a function of N variables.

2. HIERARCHICAL RENORMALIZATION GROUP

The hierarchical renormalization group is a block spin renormalization group for so called hierarchical spin models: spin models with a non-translation

invariant kinetic term, designed to make the renormalization group local. The hierarchical model and the full model belong to different universality classes. Nevertheless they are related. The hierarchical renormalization group is an ultralocal approximation to a lattice block spin renormalization group. See refs. 10, 15 and references therein. Furthermore, the hierarchical model is presumably a zeroth approximation to the full model in a lattice derivative expansion. Hierarchical results are to our experience reasonable approximations. However concerning their predictions about full models, high precision cannot be expected. For instance, ν comes out as 0.649 at the $N=1$ Ising fixed point, as compared to 0.63 from ε-expansion and Monte-Carlo methods. Hierarchical models are known to be extremely valuable in gaining qualitative information about the model under investigation and to prepare the ground for further renormalization group studies, where non-local interactions are included.

The hierarchical renormalization group for models with an N-component real scalar field is a theory of the non-linear integral transformation

$$R_{\alpha,\beta,\gamma}(Z)(\psi) = \int d\mu_\gamma(\zeta)\, Z(\beta\psi + \zeta)^\alpha \tag{3}$$

where α, β, and γ are real parameters and where $d\mu_\gamma(\zeta)$ is the Gaussian measure on $\mathbb{R}^N$ with mean zero and covariance γ (times the unit matrix); its Fourier transform is $\int d\mu_\gamma(\zeta)\, e^{i\zeta j} = e^{-(\gamma/2)\, j^2}$. We consider the renormalization group in the so called high temperature picture. See ref. 15 and references therein. Our parameter values in (3) are

$$\alpha = 2, \qquad \beta = 2^{-(2+D)/2D}, \qquad \gamma = 1 \tag{4}$$

(You may take this as the definition of the high temperature picture.) Here D is the dimension of the model. We restrict our attention to the most interesting case, when $D=3$.

The transformation (3) is the composition of three steps: taking the square, Gaussian convolution, and resealing. The numerical computation of (3) will be decomposed into these steps.

3. ALGEBRAIC FORMULATION

We use the techniques described in refs. 15 and 6. Consider first the $O(N)$-invariant case. A general even $O(N)$-invariant Boltzmann factor can be written as a sum

$$Z(\phi) = \sum_{n=0}^{\infty} (\phi^2)^n\, Z_n \tag{5}$$

and is parametrized by real coefficients Z_n. The transformation (3) becomes the following non-linear transformation in terms of the coordinates (Z_n):

$$R(Z)_n = \beta^{2n} \sum_{m=n}^{\infty} G_{n,m}(N, \gamma) \sum_{l=0}^{m} Z_l Z_{m-l} \tag{6}$$

The coefficients $G_{n,m}(N, \gamma)$ are defined as follows. (We call them structure coefficients.) Let $P_m(\phi) = (\phi^2)^m$. The Gaussian convolution of it is a linear combination

$$\int d\mu_\gamma(\zeta)\, P_m(\psi + \zeta) = \sum_{n=0}^{m} P_n(\psi)\, G_{n,m}(N, \gamma) \tag{7}$$

with coefficients (essentially the number of contractions)

$$G_{n,m}(N, \gamma) = \gamma^{m-n} \prod_{l=1}^{m-n} \frac{(N + 2(n+l-1))(n+l)}{l} \tag{8}$$

It is convenient to define $G_{n,m}(N, \gamma) = 0$ for $n > m$.

Analytical and numerical experience suggests the normalization $Z_n = X_n \cdot (\rho^n / \sqrt{(2n)!})$ with a suitable constant ρ. For notational simplicity, we prefer to display the formulas in the "raw" coefficients Z_n as in (5).

For practical computations, one has to truncate the transformation (6) to a finite number K of non-zero coefficients. For $n \leqslant K$, the resulting transformation is

$$R_K(Z)_n = \beta^{2n} \sum_{m=n}^{K} G_{n,m}(N, \gamma) \sum_{l=0}^{m} Z_l Z_{m-l} \tag{9}$$

In the high temperature picture, this truncation scheme is known to converge as $K \to \infty$. See ref. 11 for a detailed analysis of the one component case.

Renormalization group fixed points are approximated as stationary flows of the truncated transformation

$$R_K(Z^*)_n = Z_n^* \tag{10}$$

The most interesting datum of a renormalization group fixed point is its spectrum, from which one learns the behavior of the linearized flow in its vicinity. The linearized flows around a fixed point is given by

$$\mathrm{D}R_K(Y)_n = 2\beta^{2n} \sum_{m=n}^{K} G_{n,m}(N, \gamma) \sum_{l=0}^{m} Z_l^* Y_{m-l} \tag{11}$$

The spectrum of Z^* is the set of eigenvalues $\lambda^{(i)}$ of (11). The eigenvalues again are directly related to the critical exponents $\sigma^{(i)}$. See ref. 18. In our model, the-relation is $\lambda^{(i)} = \alpha^{\sigma(i)/D}$. (A peculiarity of hierarchical models is that the critical exponents are L-dependent.) The eigenvalue problem reads

$$\mathrm{D}R_K(Y^{(i)})_n = \lambda^{(i)} Y_n^{(i)} \tag{12}$$

The linearized renormalization group transformation (6) can be brought to a manifestly symmetric form in the no-truncation limit, and is thus diagonalizable. In practice, (12) turns out to be a very reliable way to determine the spectrum of renormalization group fixed points.

4. NUMERICAL RESULTS: $O(N)$ FIXED POINT

We have searched for fixed points of the system of algebraic equations (9) with a Newton algorithm. The program was written in C++ using the data type `long double` representation for real numbers. In order to check roundoff errors, we compared the `long double` with the (simple) `double` representation and found no significant deviations.

Table 1 shows the effect of truncation on the first few eigenvalues of the non-trivial $O(N)$ fixed point with $N = 2$ components in $D = 3$ dimensions. (A trivial "volume eigenvalue" $\lambda^{(0)} = 2$ has been omitted.) We see a rapid increase of accuracy with the number of couplings. With the eigenvalues decreasing we see a loss in their precision (cf. Fig. 1). To get the

Table 1. Spectrum of $O(2)_3$ Fixed Point

K	$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$
10	1.33	0.73	0.31	0.10
12	1.36	0.79	0.38	0.15
14	1.37	0.83	0.43	0.19
16	1.384	0.849	0.46	0.22
18	1.3854	0.854	0.484	0.25
20	1.3856	0.8560	0.489	0.262
22	1.38573	0.8562	0.4914	0.267
24	1.385742	0.85633	0.4917	0.268
26	1.3857434	0.8563400	0.49181	0.2691
28	1.38574348	0.85634081	0.491821	0.26923
30	1.385743489	0.85634089	0.4918225	0.269247
32	1.38574349013	0.8563409057	0.49182272	0.2692491
34	1.385743490193	0.85634090651	0.491822739	0.26924937
36	1.3857434901972	0.856340906576	0.4918227412	0.2692493996

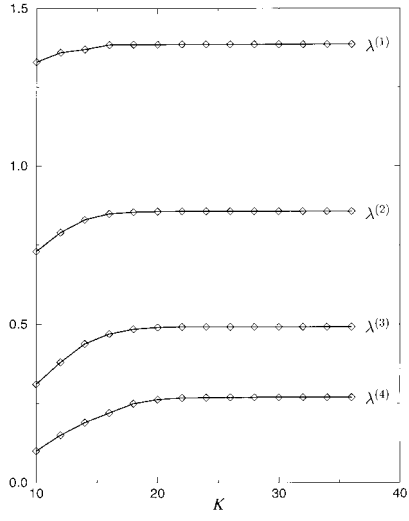

Fig. 1. Spectrum of $O(2)_3$ fixed point as function of truncation order K.

same absolute precision it seems that the smaller the eigenvalue of interest is, the bigger the number of couplings has to be chosen.

The first eigenvalue $\lambda^{(1)}$ belongs to the critical exponent $\nu = 1/\sigma^{(1)}$, or $\nu = \log(\alpha)/D \log(\lambda)$. The numerical value of ν is thus

$$\nu = 0.7082249 \tag{13}$$

With twenty couplings, the first eigenvalue comes out with an accuracy of more than three digits, with thirty couplings of more than nine digits.

A computer assisted proof for the one component case was developed in ref. 11. From it, one gets two sided bounds on the critical exponents with arbitrary precision. It confirms the accuracy of our calculation by a comparison at $N = 1$.

Table 2 shows the spectrum of the non-trivial three dimensional $O(N)$-fixed point as a function of the number of components N at truncation order $K = 30$.

Analytic continuation in N is naturally possible for the system of algebraic equations (9), since the N-dependence is encoded entirely in the structure constants (7). These depend polynomially on N. We admit that this continuation is not unique. To maintain the same precision as N increases, one has to increase K. Up to $N = 3$, the truncation order $K = 30$ suffices.

Table 2. Spectrum of $O(N)_3$ Fixed Point at $K=30$

N	$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	$\lambda^{(4)}$
−2.2	1.59824229	0.848707935	0.435463969	0.213609627
−2.0	1.58740105	0.849947302	0.437978578	0.215734559
−1.4	1.55436959	0.853424303	0.445855470	0.222533363
−0.6	1.50993452	0.857177559	0.457002524	0.232579287
−0.2	1.48805474	0.858501342	0.462742151	0.237983202
0.2	1.466769838	0.859352758	0.468493388	0.243587863
0.6	1.446379983	0.859668728	0.474160525	0.249329758
1.0	1.427172478	0.859411649	0.479637300	0.255127880
1.4	1.409390602	0.858578978	0.484816590	0.260888802
1.6	1.401091348	0.857956757	0.487263833	0.263724669
2.0	1.385743489	0.856340897	0.491822589	0.269247586
2.6	1.365884074	0.853161847	0.497720103	0.277002053
3.0	1.354668292	0.850694212	0.500989433	0.281718313
3.2	1.349623859	0.849396273	0.502426577	0.283920286

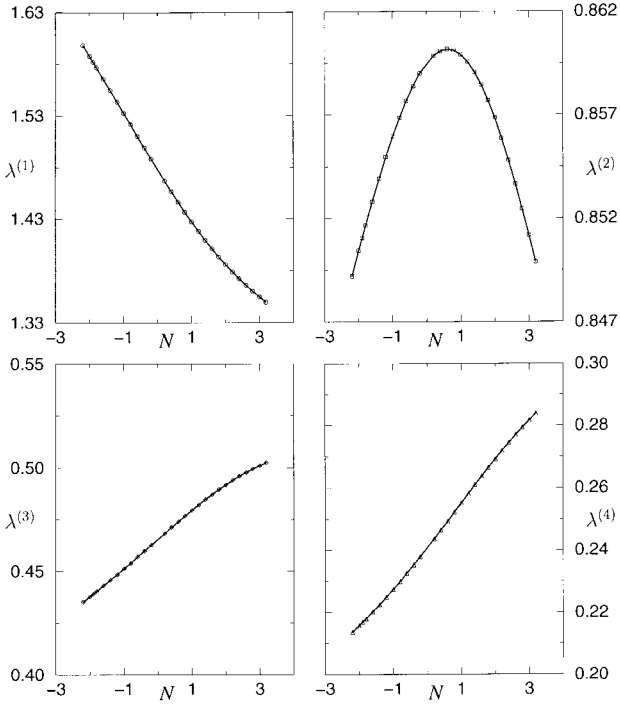

Fig. 2. Spectrum of $O(N)_3$ fixed point at $K=30$.

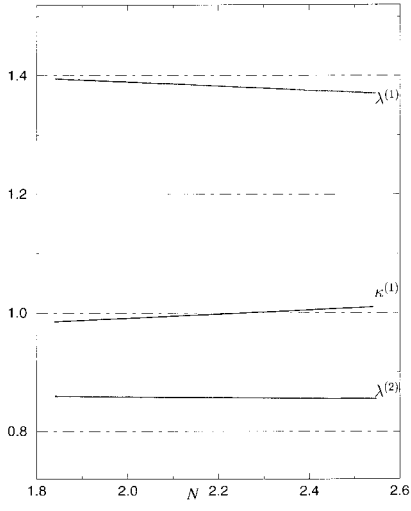

Fig. 3. Visualization of Table 3.

We rediscover among other things the well known result that the theory becomes trivial at $N = -2$, i.e., $\nu = 0.5$.

5. CUBIC INVARIANCE

Cubic symmetry is the following. Consider an N-component model. (The local spin takes values in $\mathbb{R}^N$.) The cubic symmetry is the finite group of transformations, consisting of rotations and reflections, which leaves invariant the cube $[-1, 1]^N$.

We will restrict our attention to even Boltzmann factors, with the property that

$$Z((-1)^{\sigma_1}\phi_1, ..., (-1)^{\sigma_N}\phi_N) = Z(\phi_1, ..., \phi_N) \tag{14}$$

Such Boltzmann factors are functions of $\phi_1^2, ..., \phi_N^2$. To be cubic invariant, they additionally have to be symmetric functions of $\phi_1^2, ..., \phi_N^2$. An even cubic invariant Boltzmann factor is thus a function (14), which in addition satisfies

$$Z(\phi_{\pi(1)}, ..., \phi_{\pi(N)}) = Z(\phi_1, ..., \phi_N) \tag{15}$$

for all permutations $\pi \in \mathfrak{S}_N$. We begin with the simplest case, the study of cubic perturbations of the $O(N)$-invariant fixed point.

5.1. Cubic Perturbations of the $O(N)$ Fixed Point

We can extend the polynomial basis $P_n(\phi)$ of $O(N)$-invariants as follows. Let $\underline{n} := (n_o, n_c) \in \mathbb{N} \times \{0, 1\}$ and define

$$P_{\underline{n}}(\phi) = \left(\sum_{a=1}^{N} \phi_a^2 \right)^{n_o} \left(\sum_{a=1}^{N} \phi_a^4 \right)^{n_c} \tag{16}$$

These symmetric functions do not close under multiplication. They do however close under Gaussian convolution. The following holds true:

$$\int d\mu(\zeta)\, P_{\underline{m}}(\psi + \zeta) = \sum_{\underline{n}} P_{\underline{m}}(\phi)\, G_{\underline{n}, \underline{m}}(N, \gamma) \tag{17}$$

with structure coefficients

$$\begin{aligned} G_{(n,0),(m,1)}(N,\gamma) &= N G_{0,2}(1,\gamma)\, G_{n,m}(N+4,\gamma) \\ &\quad + G_{1,2}(1,\gamma)\, G_{n-1,m}(N+6,\gamma) \\ G_{(n,1),(m,1)}(N,\gamma) &= G_{2,2}(1,\gamma)\, G_{n,m}(N+8,\gamma) \\ G_{(n,0),(m,0)}(N,\gamma) &= G_{n,m}(N,\gamma) \end{aligned} \tag{18}$$

where the $G_{n,m}(N,\gamma)$ are given by (8). The linearized renormalization group at the $O(N)$ fixed point with cubic ϕ^4-perturbations complicates to

$$\begin{aligned} &DR_K(Y)_{(n_o,n_c)} \\ &= 2\beta^{2n_o+4n_c} \sum_{m_o=n_o}^{K} \sum_{m_c=0}^{1} G_{(n_o,n_c),(m_o,m_c)}(N,\gamma) \sum_{l_o=0}^{m_o} Z^*_{(l_o,0)}\, Y_{(m_o-l_o,m_c)} \end{aligned} \tag{19}$$

The $O(N)$-invariants, defined by $n_c = 0$, form an invariant subspace. The cubic invariant eigenvectors generally have non-vanishing $O(N)$-components. The cubic eigenvalues will be denoted by $\kappa^{(i)}$. We order them according to their degree of relevance.

Table 3 shows the $O(N)$-invariant and the cubic invariant spectrum at the $O(N)$-invariant non-trivial fixed point in three dimension as a function of the number of components N. The leading cubic eigenvector is—in ε-expansion a deformation of—a cubic ϕ^4-vertex. Here we restrict our attention to eigenvectors of the ϕ^4-type times powers of ϕ^2. (The subleading eigenvalue $\kappa^{(2)}$ belonging to the cubic ϕ^6-interaction is not displayed here.)

The largest cubic eigenvector, $\kappa^{(1)}$, becomes one at a critical value N_c of N. We learn from Table 3 that $2.20 < N_c < 2.25$. A closer look at the

Table 3. Cubic Spectrum of $O(N)_3$ Fixed Point at $K = 30$

N	$\lambda^{(1)}$	$\kappa^{(1)}$	$\lambda^{(2)}$	$\kappa^{(3)}$	$\lambda^{(3)}$
1.80	1.39321	0.98503	0.85721	0.54062	0.48960
1.90	1.38942	0.98865	0.85679	0.54359	0.49072
2.00	1.38574	0.99224	0.85634	0.54656	0.49182
2.10	1.38217	0.99580	0.85588	0.54950	0.49288
2.15	1.38042	0.99757	0.85562	0.55097	0.49340
2.20	1.37870	0.99932	0.85537	0.55243	0.49391
2.25	1.37701	1.00107	0.85511	0.55389	0.49442
2.30	1.37534	1.00280	0.85484	0.55534	0.49491
2.40	1.37209	1.00624	0.85430	0.55823	0.49588
2.50	1.36893	1.00965	0.85374	0.56109	0.49681

Table 4. N_c at the $O(N)$ Fixed Point

N	K	$\kappa^{(1)}$
2.219	30	0.999**98479312540**
	40	0.999**98479665524**
2.2194	30	0.99999**876426610**
	40	0.99999**876780069**
2.21943	30	0.999999**81395124**
	40	0.999999**81748619**
2.219435	30	0.99999999**8889842**
	40	0.99999999**243344**
	50	0.99999999**243345**
2.219536	30	1.00000000**2222169**
	40	1.00000000**2575671**
	50	1.00000000**2575672**
2.21944	30	1.000000**16384551**
	40	1.000000**16738059**
2.2195	30	1.00000**226320303**
	40	1.00000**226673882**
2.220	30	1.00001**973231681**
	40	1.00001**973585855**

vicinity of N_c yields the following. Table 4 shows that the value of N_c is located inbetween

$$2.219435 < N_c < 2.219436 \tag{20}$$

(Numerical errors are negligible.) We learn furthermore the important lesson that the N-dependence of $\kappa^{(1)}$ is rather weak between the two and three component models.

6. CUBIC INVARIANT FIXED POINT

Besides the $O(N)$-invariant fixed point, we find a cubic invariant fixed point. Again we restrict our attention to the case of three dimensions. The cubic perturbations (16) have to be enlarged to a generating system of cubic invariant polynomials. We have investigated several possibilities.

6.1. Lifted Representation

The first possibility uses an over complete system. Let $\underline{n} = (n_1, n_2, n_3, \ldots) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots$. Define

$$P_{\underline{n}}(\phi) = \left(\sum_{a=1}^{N} \phi_a^2 \right)^{n_1} \left(\sum_{a=1}^{N} \phi_a^4 \right)^{n_2} \left(\sum_{a=1}^{N} \phi_a^6 \right)^{n_3} \cdots \tag{21}$$

We represent our fixed point by coordinates $Z_{\underline{n}}$. With each collection is associated a function

$$Z(\phi) = \sum_{\|\underline{n}\| \leqslant K} P_{\underline{n}}(\phi)\, Z_{\underline{n}} \tag{22}$$

To define a suitable truncation, we introduce the norm $\|n\| := n_1 + 2n_2 + 3n_3 \ldots$. In other words, we truncate the model to a maximal power of fields. The summation is restricted to the finite subset of $\mathbb{N}^\infty$ given by $\|n\| \leqslant K$.

Unfortunately, the functions (21) are not linearly independent. Moreover, the linear dependencies vary with N. Therefore, the representation (22) is not unique. As an illustration, the situation for $N = 2$ is studied in detail in the Appendix.

As we intend to use the over complete representation, we have to specify a lift of the renormalization group. We do this as follows. We have that

$$P_{\underline{n}}(\phi)\, P_{\underline{m}}(\phi) = P_{\underline{n}+\underline{m}}(\phi) \tag{23}$$

The simplicity of this operation is considered one advantage of system (21). Gaussian convolution can be written in the form

$$\int d\mu_\gamma(\zeta)\, P_{\underline{m}}(\psi+\zeta) = \sum_{\|\underline{n}\| \leqslant K} P_{\underline{n}}(\psi)\, G_{\underline{n},\underline{m}}(N,\gamma) \tag{24}$$

where the coefficients are the following. We compute

$$\sum_{a=1}^{D} \frac{\partial^2}{\partial \phi_a^2} P_{\underline{m}}(\phi) = \sum_{\underline{n} \leqslant \underline{m}} P_{\underline{n}}(\phi)\, \mathscr{G}_{\underline{n},\underline{m}} \tag{25}$$

with structure coefficients given by

$$\mathscr{G}_{\underline{n},\underline{m}} = 4abm_a m_b \quad \Leftrightarrow \quad \exists a \neq b: \begin{cases} m_a - 1 = n_a \\ m_b - 1 = n_b \\ m_{a+b-1} + 1 = n_{a+b-1} \\ \forall c \neq a, b: m_c = n_c \end{cases} \tag{26}$$

$$\mathscr{G}_{\underline{n},\underline{m}} = 4a^2 m_a (m_a - 1) \quad \Leftrightarrow \quad \exists a \neq 1: \begin{cases} m_a - 2 = n_a \\ m_{2a-1} + 1 = n_{2a-1} \\ \forall b \neq a: m_b = n_b \end{cases} \tag{27}$$

$$\mathscr{G}_{\underline{n},\underline{m}} = 2a(2a-1)\, m_a \quad \Leftrightarrow \quad \exists a \neq 1: \begin{cases} m_a - 1 = n_a \\ m_{a-1} + 1 = n_{a-1} \\ \forall b \neq a: m_b = n_b \end{cases} \tag{28}$$

$$\mathscr{G}_{\underline{n},\underline{m}} = 2(2(m_1 - 1) + N)\, m_1 \quad \Leftrightarrow \quad \begin{cases} m_1 - 1 = n_1 \\ \forall a \neq 1: m_a = n_a \end{cases} \tag{29}$$

We then compute the Gaussian integral as the matrix exponential thereof. The result is

$$G_{\underline{n},\underline{m}}(N,\gamma) = \exp\left(\frac{\gamma}{2}\mathscr{G}\right)_{\underline{n},\underline{m}} \tag{30}$$

The matrix $\mathscr{G}$ becomes upper triangular, when the couplings are sorted according to their total power of fields $\|\underline{n}\|$. (When the renormalization group is truncated to a finite power of fields, the Gaussian convolution alone does not generate higher powers.) As the matrix $\mathscr{G}$ is nilpotent, the matrix exponential is a finite sum. Our first system of equations is

$$R_K(Z)_{\underline{n}} = \beta^{2\|\underline{n}\|} \sum_{\|\underline{m}\| \leqslant K} G_{\underline{n},\underline{m}}(N,\gamma) \sum_{\underline{l}+\underline{k}=\underline{m}} Z_{\underline{l}} Z_{\underline{k}} \tag{31}$$

This system of equations defines a lifting of the renormalization group. Every fixed point of (31) becomes through (22) a fixed point of the original hierarchical renormalization group. Their spectra of eigenvalues coincide (in the no-truncation limit), but some eigenvalues become degenerate. The eigenvalue problem is completely analogous to the $O(N)$-invariant case. We do not write down the equations here.

We can use (31) to determine the cubic fixed point for any value of N. The reason is again that N enters polynomially in the structure coefficients (30). The drawback of this representation is that the number of couplings increases very fast with the order of truncation. Table 5 shows this number of couplings with $\|\underline{n}\| \leqslant K$ for $K = 1 \cdots 29$.

We looked at the system up to $K = 18$. In this model one has to compute 1597 couplings. To get an idea of the achieved precision, we did one run with $K = 19$, which means 2087 couplings at $N = 2$. (Has anyone ever computed a renormalization group with more unknowns?) The result is displayed in Table 7. For the eigenvalues $\mu^{(1)}$ an $\mu^{(3)}$ we expect an accuracy of 2 digits from our runs with $K = 18$ (Table 6). For $\mu^{(2)}$, the eigenvalue we are mainly interested in, one probably has one additional digit.

As in the $O(N)$ case we recognize one eigenvalue to become marginal at a critical number of components $N_c^{(\mathrm{cub})}$. The value of $N_c^{(\mathrm{cub})}$ comes out as

$$2.2 \lesssim N_c^{(\mathrm{cub})} \lesssim 2.25 \tag{32}$$

Table 7 combined with the investigation in polar coordinates for the $N = 2$-model, to be described below, suggests that the precision of $\mu^{(2)}$ at $K = 18$ is three digits.

Table 5. Number of Couplings in the Lifted System for Given Truncation *K*

K	#coups	K	#coups	K	#coups
1	2	10	139	20	2714
2	3	11	195	21	3506
2	4	12	272	22	4508
3	7	13	373	23	5763
4	12	14	508	24	7338
5	19	15	684	25	9296
6	30	16	915	26	11732
7	45	17	1212	27	14742
8	67	18	1597	28	18460
9	97	19	2087	29	23025

Table 6. Cubic Spectrum at $K = 18$

N	$\mu^{(1)}$	$\mu^{(2)}$	$\mu^{(3)}$	$\mu^{(4)}$	$\mu^{(5)}$	$\mu^{(5')}$
2.00	1.4337	1.0188	0.8540	0.5968	0.4887	0.4697
2.10	1.3988	1.0066	0.8551	0.5712	0.5367	0.4759
2.15	1.3857	1.0034	0.8540	0.5717	0.5545	0.4811
2.20	1.3790	1.0013	0.8531	0.5963	0.5500	0.4840
2.25	1.3765	0.9996	0.8525	0.6163	0.5481	0.4844
2.30	1.3758	0.9980	0.8521	0.6309	0.5480	0.4836
2.40	1.3761	0.9952	0.8515	0.6498	0.5491	0.4809
2.50	1.3766	0.9926	0.8508	0.6606	0.5498	0.4783
2.70	1.3765	0.9880	0.8490	0.6704	0.5489	0.4740
3.00	1.3735	0.9813	0.8451	0.6730	0.5442	0.4685

The calculation with $K = 19$ was only performed for the case $N = 2$ and $D = 3$, as it required about 22 hours CPU time on an IBM RS/6000-590 workstation (121 SPECint92), $K = 18$ required around 10 hours on the same system.

As the value $\mu^{(2)}$ decreases with increasing K, we expect the critical value $N_c^{(\mathrm{cub})}$ to be given an upper bound by the value $N_c^{(\mathrm{cub})} < 2.25$.

For the Newton iteration to converge, one has to start from a sufficiently close initial guess. We refrain from presenting a lengthy table of fixed point couplings. Instead, we present a sufficiently accurate initial guess from which anyone can reconstruct the cubic fixed point. (Later we will discuss the fixed point couplings in the special case when $N = 2$.) Our search strategy was the following. We started at two ends of the interval of

Table 7. The First Six Eigenvalues of the Spectrum at the Cubical Fixed Point, $N = 2$, for Three Different Truncation Orders[a]

K	$\mu^{(1)}$	$\mu^{(2)}$	$\mu^{(3)}$
15	1.488	1.026	0.821
16	1.4557	1.0224	0.8399
18	1.43377	1.01889	0.85407
19	1.430341	1.018439	0.856580

[a] $m_{\max} = 4$ in all cases. For comparison the last column shows the results from the calculation with the over determined basis, truncation order $K = 18$, cf. Tables 5 and 6.

interest, at $N=2$ and at $N=3$. First we looked for the $O(N)$ fixed point, using the start values

$$Z_{(0,0,0,0,\ldots)} = 0.3$$
$$Z_{(1,0,0,0,\ldots)} = 0.06$$
$$Z_{(2,0,0,0,\ldots)} = 0.001$$
$$Z_{(n,0,0,0,\ldots)} = 0 \;\forall n > 2$$

Within our scheme (no rescaling on the $Z_{\underline{n}}$) this initial guess turned out to be sufficient to converge to the the $O(N)$ fixed point at both values of N. To find the cubic fixed point, we took the $O(N)$ fixed point and changed the following coefficients:

$$Z_{(0,0,0,0,\ldots)} = 0.64$$
$$Z_{(0,1,0,0,\ldots)} = 0.007$$
$$Z_{(0,0,1,0,\ldots)} = 0.00011$$

for the case $N=2$ and

$$Z_{(0,0,0,0,\ldots)} = 0.46$$
$$Z_{(0,1,0,0,\ldots)} = 0.007$$
$$Z_{(0,0,1,0,\ldots)} = 0.00011$$

for $N=3$. (These values should be good enogh for $K \geqslant 17$.) Having found the cubic fixed point at these two values of N we stepped towards the critical N_c from both sides, using the last fixed point as starting vector for the next iteration.

6.2. $N=2$ Using Polar Coordinates

For fixed N, it seems natural to introduce spherical coordinates. For $N=2$, we have that

$$Z(\phi) = Z(r, \varphi) = \sum_{m \geqslant 0} Z_m(r) \cos(m\varphi) \tag{33}$$

The cubic symmetry can be implemented by requiring that the coefficients $Z_m(r)$ be zero for all m which are not integer multiples of 4. The result Z'

of a hierarchical RG transformation applied to Z can be written in the form

$$Z'(\phi) = e^{-(1/2)\,\beta^2\phi^2} \int d^2\zeta\; e^{-(1/2)\,\zeta^2} e^{\beta\phi\zeta} Z^2(\zeta) \tag{34}$$

We introduce polar coordinates for ζ and ϕ through

$$\zeta = r(\cos\varphi, \sin\varphi), \qquad \phi = R(\cos\theta, \sin\theta) \tag{35}$$

and expand the square of Z

$$Z^2(\zeta) = \sum_{m \geqslant 0} (Z^2)_m(r) \cos(m\zeta) \tag{36}$$

The $(Z^2)_m$ are related to the Z_m through

$$(Z^2)_m = \tfrac{1}{2} \sum_{m_1} \sum_{m_2} Z_{m_1} Z_{m_2} (\delta_{m,\, m_1 + m_2} + \delta_{m,\, |m_1 - m_2|}) \tag{37}$$

The angular integration in Eq. (34) can be performed, resulting in

$$Z'(R, \theta) = \sum_m Z'_m(R) \cos(m\theta) \tag{38}$$

with

$$Z'_m(R) = e^{-(1/2)\,\beta^2 R^2} \int_0^\infty dr\; r\, e^{-(1/2)\,r^2} I_m(\beta r R) (Z^2)_m(r) \tag{39}$$

Here, I_m denotes the modified Bessel-function of order m. The next step is an expansion of the Z_m in a power series of r^2,

$$Z_m = \sum_l Z_{ml} r^{2l} \tag{40}$$

Expanding the Bessel-function and performing the integration over r yields the following relation (for m even):

$$Z'_{ml'} = \sum_l C^m_{l'l} (Z^2)_{ml} \tag{41}$$

with

$$C^m_{l'l} = 2^{l-l'} \beta^{2l'} \sum_{j=0}^{l' - m/2} (-1)^j \frac{(l + l' - j)!}{(l' - (m/2) - j)!\; (l' + (m/2) - j)!\; j!} \tag{42}$$

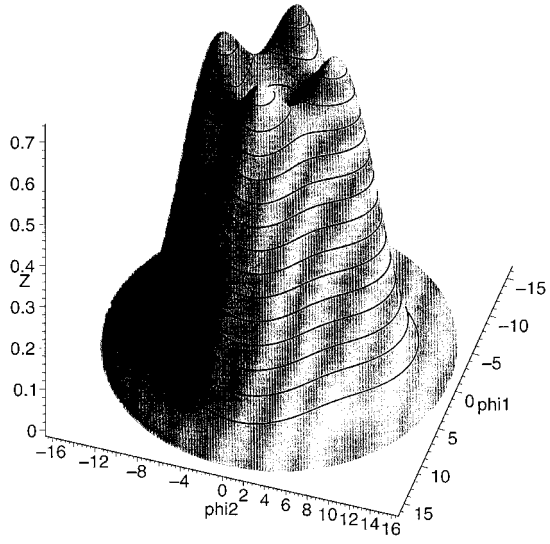

Fig. 4. Cubic fixed point Boltzmann factor Z^* for $N=2$, as function of the two spin components ϕ_1 and ϕ_2.

A combination of the transformation with the square operation Eq. (37) yields the structure coefficients of the complete hierarchical RG transformation for $N=2$. The matrix representing the linearized transformation is then easy to compute.

A Newton solver was used to find the cubic fixed point. It is known[1] that for $N=2$ the cubical fixed point can be exactly mapped (by a rotation of the spin vector with an angle $\pi/4$) to the product of two independent Ising ($N=1$) fixed points. Our fixed point, plotted in Fig. 4, is indeed of this type.

We determined the eigenvalues of the linearized transformation at that fixed point. Some care was devoted again to study the truncation effects. Table 8 shows the six leading eigenvalues at the cubical fixed points for three different truncation orders. To the given precision, it is sufficient to include $K=32$ powers of r^2 in the ansatz. Furthermore, it is sufficient to include "angular momenta" m with $m \leqslant m_{\max}=4$. Going to $m_{\max}=5$ did not change the results for the exponents at all. From the factorization of the cubical fixed point into two Ising fixed points it follows that the spectrum can be built from the Ising spectrum through

$$\lambda_{i,\,\mathrm{cub}} = \tfrac{1}{2}\lambda_{j,\,\mathrm{Isi}}\,\lambda_{k,\,\mathrm{Isi}} \tag{43}$$

Table 8. Coupling Constants of the Cubical Fixed Point, for $K = 30$ and $m_{max} = 4$

k	$K = 30$	$K = 26$	$K = 20$	$K = 19$	$K = 18$	μ
1	1.42717	1.42718	1.42867	1.43034	1.43377	1.4337
2	1.01841	1.01840	1.01831	1.01844	1.01889	1.0188
3	0.859411	0.859393	0.857925	0.856580	0.854073	0.8540
4	0.613255	0.613058	0.605915	0.602012	0.596873	0.5968
5	0.479634	0.479563	0.476318	0.473871	0.469780	0.4887
6	0.369252	0.368630	0.360018	0.357220	0.353718	0.4697

Table 9

l	Z_{0l}	Z_{1l}	Z_{2l}	Z_{3l}	Z_{4l}
1	0.6404E − 01	0.0000E + 01	0.0000E + 01	0.0000E + 01	0.0000E + 01
2	0.3162E − 02	0.1518E − 03	0.0000E + 01	0.0000E + 01	0.0000E + 01
3	0.9355E − 04	0.1190E − 04	0.0000E + 01	0.0000E + 01	0.0000E + 01
4	0.1902E − 05	0.4309E − 06	0.3827E − 08	0.0000E + 01	0.0000E + 01
5	0.2877E − 07	0.9721E − 08	0.2351E − 09	0.0000E + 01	0.0000E + 01
6	0.3409E − 09	0.1550E − 09	0.6851E − 11	0.3149E − 13	0.0000E + 01
7	0.3280E − 11	0.1877E − 11	0.1269E − 12	0.1624E − 14	0.0000E + 01
8	0.2633E − 13	0.1807E − 13	0.1869E − 14	0.4026E − 16	0.1135E − 18
9	0.1799E − 15	0.1430E − 15	0.1728E − 16	0.6414E − 18	0.5103E − 20
10	0.1064E − 17	0.9518E − 18	0.1420E − 18	0.7407E − 20	0.1111E − 21
11	0.5515E − 20	0.5436E − 20	0.9673E − 21	0.6629E − 22	0.1567E − 23
12	0.2532E − 22	0.2703E − 22	0.5586E − 23	0.4798E − 24	0.1610E − 25
13	0.1039E − 24	0.1185E − 24	0.2783E − 25	0.2895E − 26	0.1290E − 27
14	0.3836E − 27	0.4624E − 27	0.1214E − 27	0.1488E − 28	0.8391E − 30
15	0.1283E − 29	0.1620E − 29	0.4688E − 30	0.6628E − 31	0.4568E − 32
16	0.3911E − 32	0.5135E − 32	0.1618E − 32	0.2593E − 33	0.2125E − 34
17	0.1090E − 34	0.1479E − 34	0.5024E − 35	0.9000E − 36	0.8583E − 37
18	0.2786E − 37	0.3890E − 37	0.1412E − 37	0.2791E − 38	0.3042E − 39
19	0.6533E − 40	0.9346E − 40	0.3596E − 40	0.7760E − 41	0.9517E − 42
20	0.1401E − 42	0.2046E − 42	0.8292E − 43	0.1934E − 43	0.2633E − 44
21	0.2728E − 45	0.4055E − 45	0.1719E − 45	0.4294E − 46	0.6415E − 47
22	0.4759E − 48	0.7182E − 48	0.3167E − 48	0.8407E − 49	0.1363E − 49
23	0.7311E − 51	0.1117E − 50	0.5100E − 51	0.1428E − 51	0.2490E − 52
24	0.9663E − 54	0.1493E − 53	0.7021E − 54	0.2061E − 54	0.3832E − 55
25	0.1068E − 56	0.1666E − 56	0.8042E − 57	0.2461E − 57	0.4847E − 58
26	0.9536E − 60	0.1499E − 59	0.7407E − 60	0.2353E − 60	0.4879E − 61
27	0.6560E − 63	0.1039E − 62	0.5239E − 63	0.1721E − 63	0.3738E − 64
28	0.3246E − 66	0.5174E − 66	0.2658E − 66	0.8996E − 67	0.2039E − 67
29	0.1023E − 69	0.1641E − 69	0.8570E − 70	0.2982E − 70	0.7026E − 71
30	0.1539E − 73	0.2479E − 73	0.1315E − 73	0.4694E − 74	0.1146E − 74

From Table 2 we read the leading Ising eigenvalue being 1.4271725.[15] Its square divided by two is 1.0184107, in nice agreement with the result for the subleading eigenvalue of the cubic fixed point. The whole Ising spectrum is part of the Cubic spectrum as follows when one of the Ising eigenvalues is taken equal to the volume eigenvalue two.

The comparison reveals that the spectrum in the polar coordinate representation and the spectrum in the overdetermined representation agree. The value $K=18$ does not bring about a very high degree of accuracy for the cubic invariant fixed point in the overdetermined representation. But our computer resources did not admit larger values of K. We learn from the comparison that the overdetermined representation generates new spurious eigenvalues. They should converge to the true ones in the no-truncation limit. The first such spurious eigenvalue is however a subleading (or irrelevant) one.

It might be interesting to have a look at the fixed point couplings themselves. They are presented for $(K, m_{\max})=(30, 4)$ in Table 9.

7. CONCLUSIONS

In the framework of the hierarchical renormalization group, we have studied the stability of both the $O(N)$ symmetric and the cubical fixed point for $D=3$, in the range between $N=2$ and $N=3$. Concerning the $O(N)$ invariant fixed point, our analysis confirms the expected picture but with a significantly shifted critical component number N_c. While the value of N_c is very close to three in the full model, the hierarchical $O(N)$ fixed point in three dimensions becomes unstable with respect to cubic perturbations already at $N_c=2.219$. It has been observed in other contexts that the dependence of certain quantities on the number of spin components is shifted towards smaller values of N in the hierarchical approximation. For instance, the two dimensional hierarchical non-linear σ-model is asymptotically free for $N>1$ as opposed to $N>2$ in the full setting.[5, 16] In the σ-model, this is due to the absence of wave function renormalization in the hierarchical approximation. Concerning the cubical fixed point, the issue of the basis of interactions and its continuation to non-integer values of N turns out to be already nontrivial. We solved this problem by using an over-complete set of functions. The big number of couplings in this approach required a considerable effort to solve the fixed point equations. Furthermore, it might be considered as a problem that the continuation from integer N to the real domain is not unique. The cubic invariant fixed point also posesses a critical component number $N_c^{(\mathrm{cub})}$. But the corresponding eigenperturbation becomes irrelevant above $N_c^{(\mathrm{cub})}$ and relevant below $N_c^{(\mathrm{cub})}$. This is opposite to the $O(N)$-invariant fixed point. Within

the accuracy of our computations, the values of N_c and $N_c^{(\mathrm{cub})}$ coincide. A heuristic explanation of this could be the existence of a renormalized cubic $(\phi^2)^2$-trajectory inbetween both fixed points in the critical surface, on which the flow reverses its direction at the critical component number. An investigation of this conjectured trajectory would be very interesting, in particular the scenario exactly at N_c.

Supplementing the present work by a high order ε-expansion (which is certainly feasible) would be interesting and useful.

APPENDIX: A BASIS FOR THE $N=2$ CASE

The polynomials (22) do not form a basis for cubic invariant functions. There are linear dependencies. The Simplest relation among these is given by

$$P_{(0,2,0,0)} = 2P_{(0,0,0,1)} - 2P_{(2,1,0,0)} + P_{(4,0,0,0)} \tag{44}$$

where the indices are to be understood in the sense of (21) with all n_i for $i>4$ are equal to zero.

For the case $N=2$ a basis for polynomials of type (22) is given by

$$P_{(n_o,n_c)}(\phi) := \left(\sum_{a=1}^{2} \phi_a^2\right)^{n_o} \sum_{a=1}^{2} \phi^{4n_c} \tag{45}$$

The relation with (21) is that (45) equals $P_{\underline{m}}$ with $m_1 = n_o$, $m_{4n_c} = 1$, and all others are zero.

This basis has the following complications. A product of two such polynomials is no more the polynomial to the index given by the sum of the indices of the factors. In general, a product of two basis elements is a linear combination

$$P_{(n_o,n_c)}(\phi)\, P_{(m_o,m_c)}(\phi) = \sum_{\substack{(k_o,k_c)\\ k_o=n_o+m_o\\ k_c=n_c+m_c}} C_{(k_o,k_c)}^{(n_o,n_c),(m_o,m_c)} P_{(k_o,k_c)}(\phi) \tag{46}$$

We do not have closed expressions for the coefficients $C_{(k_o,k_c)}^{(n_o,n_c),(m_o,m_c)}$. We tabulated those we needed in our programs. The table was generated using computer algebra.

The Gaussian integration of polynomials of type (45) is given by

$$\begin{aligned}\int d\mu_\gamma(\zeta)\, P_{(n_o,n_c)}(\psi+\zeta) = &\sum_{k_o=0}^{n_o} \sum_{k_c=0}^{n_c} I_{k_o,k_c}^{n_o,n_c}(N,\gamma)\, P_{(k_o,k_c)}(\psi)\\ &+ \sum_{k_o=0}^{n_o} \sum_{k_c=0}^{n_c} I_{k_o,k_c+1/2}^{n_o,n_c}(N,\gamma)\, P_{(k_o,k_c+1/2)}(\psi)\end{aligned} \tag{47}$$

with

$$I^{n_o, n_c}_{k_o, k_c}(N, \gamma) = G_{2n_c, 2k_c}(1, \gamma)\, G_{n_o, k_o}(4n_c + 4k_c + N, \gamma) \tag{48}$$

where $G_{n,m}(N, \gamma)$ is as in (8). The formula (47) does not yet provide a formula for the Gaussian integration in the basis (45), as the second term is not yet expanded in this basis. Its expansion involves again rather complicated structure coefficients which we determined by means of computer algebra.

We used this representation to check our results with the overdetermined representation. This polynomial basis turned out to a less appropriate representation than the following one with polar coordinates. Both suffer the drawback that they apply to $N = 2$ only, and cannot be continued to any N. (They can be continued but do not coincide with the desired models at integer N.)

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